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Abstract: Time-delay robustness analysis for homogeneous systems with negative degree is presented in the paper. It is established that if a system is homogeneous with negative degree and asymptotically stable at its origin in the absence of delays, then stability property is preserved with respect to a compact set containing the origin for any delay value. It is shown that obtained results can also be used for locally homogeneous systems. A simulation example is included to support theoretical results.

Keywords: Homogeneous system, time-delay system, robustness.

1. INTRODUCTION

The time-delay dynamical systems (whose models are represented by functional differential equations) appear in many areas of science and technology, like e.g. systems biology and networked/distributed systems Chiasson and Loiseau (2007), Erneux (2009). Then influence of delays on the system stability and performance is critical for many natural and human-developed systems Gu et al. (2003), Hale (1977), Kolmanovsky and Nosov (1986). Synthesis of control and estimation algorithms, which are robust with respect to uncertain and time-varying delays, is an important and quickly developing topic of the modern control theory Fridman (2014). Despite of variety of methods solving this problem, most of them deal with linear time-delay models, which is originated by complexity of stability analysis for the nonlinear case (design of a Lyapunov-Krasovskii functional or a Lyapunov-Razumikhin function is a complex problem), and that constructive and computationally tractable conditions exist for linear systems only Richard (2003).

The theory of homogeneous dynamical systems has been established and well explored for ordinary differential equations Bacciotti and Rosier (2001), Bhat and Bernstein (2005), Kawski (1991), Zubov (1958), Qian et al. (2015) and differential inclusions Bernuau et al. (2014), Levant (2005). Linear systems form a subclass of homogeneous ones, moreover the main feature of a homogeneous nonlinear system is that its local behavior is the same as the global one Bacciotti and Rosier (2001), as in the linear case. In addition, the homogeneous stable/unstable systems admit homogeneous Lyapunov/Chetaev functions Zubov (1958), Rosier (1992), Efimov and Perruquetti (2010), Efimov et al. (2014a). Since the subclass of nonlinear systems, having a global behavior, is rather small, the concept of local homogeneity has been introduced Zubov (1958), Andrieu et al. (2008), Efimov and Perruquetti (2010). Attempts to apply and extend the homogeneity theory for nonlinear functional differential equations, in order to simplify their stability analysis and design, have been performed in Efimov and Perruquetti (2011), Efimov et al. (2014b), Efimov et al. (2016). Applications of the conventional homogeneity framework for analysis of time-delay systems (considering delay as a kind of perturbation, for instance) have been carried out even earlier in Aleksandrov and Zhabko (2012), Asl and Ulsoy (2000), Bokharaie et al. (2010), Diblik (1997), Mazenc et al. (2001).

In Efimov et al. (2016) it has been shown that homogeneous time-delay systems have certain stability robustness with respect to delays, e.g. if they are globally asymptotically stable for some delay, then they preserve this property independently of delay (IOD). For the case of nonnegative degree it has been proven that global asymptotic stability in the delay-free case implies local asymptotic stability for a sufficiently small delay (similarly to linear systems, i.e. a case of zero degree). The main goal of this work is to develop that result for the case of negative degree, such an extension is not trivial since the proof of Efimov et al. (2016) was heavily based on Lipschitz

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continuity property of the system, but this property is not satisfied for a negative degree. It will be shown that in the case of negative degree, if the system is globally asymptotically stable in the delay-free case, then for any delay it is globally asymptotically stable with respect to a compact set containing the origin.

The outline of this paper is as follows. The preliminary definitions and the homogeneity for time-delay systems are given in Section 2. The main result is presented in Section 3. An example is considered in Section 4.

2. PRELIMINARIES

Consider an autonomous functional differential equation of retarded type Kolmanovsky and Nosov (1986):

$$dx(t)/dt = f(x_t),\ t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ and $x_t \in C_{[-\tau,0]}$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ (we denote by $C_{[-\tau,0]}$ the Banach space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm $||\phi|| = \sup_{-\tau\leq\varsigma\leq 0} |\phi(\varsigma)|$, where $|\cdot|$ is the standard Euclidean norm); $f : C_{[-\tau,0]} \to \mathbb{R}^n$ ensures existence and uniqueness of solutions in forward time, $f(0) = 0$. The representation (1) includes pointwise or distributed time-delay systems. We assume that solutions of the system (1) satisfy the initial functional condition $x_0 \in C_{[-\tau,0]}$ for which the system (1) has a unique solution $x(t, x_0)$ and $x_{t,x_0}(s) = x(t+s, x_0)$ for $-\tau \leq s \leq 0$, which is defined on some finite time interval $[-\tau, T)$ (we will use the notation $x(t)$ to reference $x(t, x_0)$ if the origin of x_0 is clear from the context).

The upper right-hand Dini derivative of a locally Lipschitz continuous functional $V : C_{[-\tau,0]} \to \mathbb{R}_+$ along the system (1) solutions is defined as follows for any $\phi \in C_{[-\tau,0]}$:

$$D^+V(\phi) = \lim \sup_{h\to 0^+} \frac{1}{h}[V(\phi_h) - V(\phi)],$$

where $\phi_h \in C_{[-\tau,0]}$ for $0 < h < \tau$ is given by

$$\phi_h = \begin{cases} \phi(\theta + h), & \theta \in [-\tau, -h) \\ \phi(0) + f(\phi)(\theta + h), & \theta \in [-h, 0]. \end{cases}$$

For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ the upper directional Dini derivative is defined as follows:

$$D^+V[x_t(0)]f(x_t) = \lim \sup_{h\to 0^+} \frac{V[x_t(0)+hf(x_t)]-V[x_t(0)]}{h}.$$

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is decreasing to zero for any fixed $r \in \mathbb{R}_+$. The symbol $\overline{1, m}$ is used to denote a sequence of integers $1, ..., m$.

2.1 Stability definitions

Let Ω be a neighborhood of the origin in $C_{[-\tau,0]}$.

Definition 1 Moulay et al. (2008) The system (1) is said to be

(a) stable at the origin into Ω if there is $\sigma \in \mathcal{K}$ such that for any $x_0 \in \Omega$, $|x(t, x_0)| \leq \sigma(||x_0||)$ for all $t \geq 0$;

(b) asymptotically stable at the origin into Ω if it is stable into Ω and $\lim_{t\to+\infty} |x(t, x_0)| = 0$ for any $x_0 \in \Omega$;

(c) finite-time stable at the origin into Ω if it is stable into Ω and for any $x_0 \in \Omega$ there exists $0 \leq T^{x_0} < +\infty$ such that $x(t, x_0) = 0$ for all $t \geq T^{x_0}$. The functional $T_0(x_0) = \inf\{T^{x_0} \geq 0 : x(t, x_0) = 0\ \forall t \geq T^{x_0}\}$ is called the settling time of the system (1).

If $\Omega = C_{[-\tau,0]}$, then the corresponding properties are called global stability/asymptotic stability/finite-time stability.

2.2 Homogeneity

For any $r_i > 0$, $i = \overline{1, n}$ and $\lambda > 0$, define the dilation matrix $\Lambda_r(\lambda) = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$ and the vector of weights $r = [r_1, ..., r_n]^T$.

For any $r_i > 0$, $i = \overline{1, n}$ and $x \in \mathbb{R}^n$ the homogeneous norm can be defined as follows

$$|x|_r = \left(\sum_{i=1}^n |x_i|^{\rho/r_i}\right)^{1/\rho},\ \rho \geq \max_{1\leq i\leq n} r_i.$$

For all $x \in \mathbb{R}^n$, its Euclidean norm $|x|$ is related with the homogeneous one:

$$\underline{\sigma}_r(|x|_r) \leq |x| \leq \bar{\sigma}_r(|x|_r)$$

for some $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$. The homogeneous norm has an important property that is $|\Lambda_r(\lambda)x|_r = \lambda|x|_r$ for all $x \in \mathbb{R}^n$. Define $\mathbb{S}_r = \{x \in \mathbb{R}^n : |x|_r = 1\}$.

For any $r_i > 0$, $i = \overline{1, n}$ and $\phi \in C_{[-\tau,0]}$ the homogeneous norm can be defined as follows

$$||\phi||_r = \left(\sum_{i=1}^n ||\phi_i||^{\rho/r_i}\right)^{1/\rho},\ \rho \geq \max_{1\leq i\leq n} r_i.$$

There exist two functions $\underline{\rho}_r, \bar{\rho}_r \in \mathcal{K}_\infty$ such that for all $\phi \in C_{[-\tau,0]}$ Efimov et al. (2014b):

$$\underline{\rho}_r(||\phi||_r) \leq ||\phi|| \leq \bar{\rho}_r(||\phi||_r). \tag{2}$$

The homogeneous norm in the Banach space has the same important property that $||\Lambda_r(\lambda)\phi||_r = \lambda||\phi||_r$ for all $\phi \in C_{[-\tau,0]}$. In $C_{[-\tau,0]}$ the corresponding unit sphere $\mathcal{S}_r = \{\phi \in C_{[-\tau,0]} : ||\phi||_r = 1\}$. Define $B^\tau_\rho = \{\phi \in C_{[-\tau,0]} : ||\phi||_r \leq \rho\}$ as a closed ball of radius $\rho > 0$ in $C_{[-\tau,0]}$.

Definition 2 Efimov and Perruquetti (2011) The function $g : C_{[-\tau,0]} \to \mathbb{R}$ is called r-homogeneous ($r_i > 0$, $i = \overline{1, n}$), if for any $\phi \in C_{[-\tau,0]}$ the relation

$$g(\Lambda_r(\lambda)\phi) = \lambda^d g(\phi)$$

holds for some $d \in \mathbb{R}$ and all $\lambda > 0$.

The function $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is called r-homogeneous ($r_i > 0$, $i = \overline{1, n}$), if for any $\phi \in C_{[-\tau,0]}$ the relation

$$f(\Lambda_r(\lambda)\phi) = \lambda^d\Lambda_r(\lambda)f(\phi)$$

holds for some $d \geq -\min_{1\leq i\leq n} r_i$ and all $\lambda > 0$.

In both cases, the constant d is called the degree of homogeneity.

The introduced notion of weighted homogeneity in $C_{[-\tau,0]}$ is reduced to the standard one in $\mathbb{R}^n$ if $\tau \to 0$.

Lemma 1. Let $f : C_{[-\tau,0]} \to \mathbb{R}^n$ be locally bounded and r-homogeneous with degree d, then there exist $k > 0$ such that

$$||f(x)||_r \le k \max_{1\le i\le n} ||x||_r^{1+d/r_i} \quad \forall x \in C_{[-\tau,0]}.$$

Proof. Take $x \in C_{[-\tau,0]}$, then there exists $\xi \in \mathcal{S}_r$ such that $x = \Lambda_r(\lambda)\xi$ for $\lambda = ||x||_r$. By definition of homogeneity we obtain:

$$||f(x)||_r = ||f(\Lambda_r(\lambda)\xi)||_r \le \max_{1\le i\le n} \lambda^{1+d/r_i}||f(\xi)||_r$$
$$\le k \max_{1\le i\le n} ||x||_r^{1+d/r_i}$$

for $k = \sup_{\xi\in\mathcal{S}_r} ||f(\xi)||_r$. Note that $d > -\min_{1\le i\le n} r_i$ for a continuous f, then $1 + d/r_i > 0$. □

This lemma implies that if a continuous $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is r-homogeneous with degree d, then it admits a kind of Hölder continuity at the origin (i.e. $\lambda < 1$) for the homogeneous norm of exponent $1+d/\max_{1\le i\le n} r_i$ if $d \ge 0$ or $1+d/\min_{1\le i\le n} r_i$ if $d < 0$, and taking into account the properties of the functions $\underline{\rho}_r, \bar{\rho}_r$ a similar conclusion holds for the conventional norm also.

Lemma 2. Let $f : C_{[-\tau,0]} \to \mathbb{R}^n$ be r-homogeneous with degree d and uniformly continuous in B^τ_ρ [1] for some $\rho > 0$, then for any $\eta > 0$ there exists $k > 0$ such that

$$||f(x) - f(z)||_r \le \max\{k \max_{1\le i\le n} ||x - z||_r^{1+\frac{d}{r_i}}, \eta\} \ \forall x, z \in B^\tau_\rho.$$

Proof. Proof. Since f is uniformly continuous in B^τ_ρ, then for $\eta > 0$ there is $\delta_\eta > 0$ such that $||f(x) - f(z)||_r < \eta$ for all $x, z \in B^\tau_\rho$ with $||x - z||_r < \delta_\eta$. Take $e = x - z \in C_{[-\tau,0]}$, then there exists $\epsilon \in \mathcal{S}_r$ such that $e = \Lambda_r(\lambda)\epsilon$ with $\lambda = ||e||_r$. For $\lambda \ge \delta_\eta$ define $z = \Lambda_r(\lambda)\zeta$ for some $\zeta \in C_{[-\tau,0]}$, then

$$||f(x) - f(z)||_r = ||f(z + e) - f(z)||_r$$
$$= ||\lambda^d\Lambda_r(\lambda)[f(\zeta + \epsilon) - f(\zeta)]||_r$$
$$\le \max_{1\le i\le n} \lambda^{1+d/r_i}||f(\zeta + \epsilon) - f(\zeta)||_r.$$

Since $||z||_r \le \rho$, then $||\zeta||_r \le \delta_\eta^{-1}\rho$ and

$$||f(\zeta + \epsilon) - f(\zeta)||_r \le k$$

for some $k > 0$ dependent on ρ and η for any $x, z \in B^\tau_\rho$ with $||x - z||_r \ge \delta_\eta$. The claim follows by combining these both cases. □

Corollary 3. Let $f : C_{[-\tau,0]} \to \mathbb{R}^n$ be r-homogeneous with degree $d < 0$ and uniformly continuous in B^τ_ρ for some $\rho > 0$, then for any $\eta > 0$ there exist $k' > 0$ such that

$$||f(x) - f(z)||_r \le \max\{k'||x - z||_r, \eta\} \quad \forall x, z \in B^\tau_\rho.$$

Proof. The result follows Lemma 2 noticing that for $d < 0$ we have $0 < 1 + d/r_i < 1$ for all $1 \le i \le n$, then for any $\eta > 0$ and $\rho > 0$ there exists $\tilde{k} > 0$ such that

$$\max_{1\le i\le n} ||x - z||_r^{1+d/r_i} \le \max\{\tilde{k}||x - z||_r, \frac{\eta}{k}\} \quad \forall x, z \in B^\tau_\rho.$$

□

An advantage of homogeneous systems described by ordinary differential equations is that any of its solutions can be obtained from another solution under the dilation rescaling and a suitable time re-parametrization. A similar property holds for functional homogeneous systems:

Proposition 4. Efimov et al. (2014b) Let $x : \mathbb{R}_+ \to \mathbb{R}^n$ be a solution of the r-homogeneous system (1) with the degree $d = 0$ for an initial condition $x_0 \in C_{[-\tau,0]}$. For any $\lambda > 0$ define $y(t) = \Lambda_r(\lambda)x(t)$ for all $t \ge 0$, then $y(t)$ is also a solution of (1) with the initial condition $y_0 = \Lambda_r(\lambda)x_0$.

Proposition 5. Efimov et al. (2016) Let $x(t, x_0)$ be a solution of the r-homogeneous system (1) with the degree $d \ne 0$ for an initial condition $x_0 \in C_{[-\tau,0]}$, $\tau \in (0, +\infty)$. For any $\lambda > 0$ the functional differential equation

$$dy(t)/dt = f(y_t),\ t \ge 0 \qquad (3)$$

with $y_t \in C_{[-\lambda^{-d}\tau,0]}$, has a solution $y(t, y_0) = \Lambda_r(\lambda)x(\lambda^d t, x_0)$ [2] for all $t \ge 0$ with the initial condition $y_0 \in C_{[-\lambda^{-d}\tau,0]}$, $y_0(s) = \Lambda_r(\lambda)x_0(\lambda^d s)$ for $s \in [-\lambda^{-d}\tau, 0]$.

The following results have also been obtained in Efimov et al. (2016):

Lemma 6. Let the system (1) be r-homogeneous with degree $d \ne 0$ and globally asymptotically stable at the origin for some delay $0 < \tau_0 < +\infty$, then it is globally asymptotically stable at the origin IOD.

Lemma 7. Efimov et al. (2016) Let the system (1) be r-homogeneous with degree $d < 0$ and asymptotically stable at the origin into the set $B^\tau_{\rho'}$ for some $0 < \rho' < +\infty$ for any value of delay $0 \le \tau \le \tau_0$ with $0 < \tau_0 < +\infty$, then it is globally asymptotically stable at the origin IOD.

Lemma 8. Efimov et al. (2016) Let the system (1) be r-homogeneous with degree $d > 0$ and the set B^τ_ρ for some $0 < \rho < +\infty$ be uniformly globally asymptotically stable for any value of delay $0 \le \tau \le \tau_0 < +\infty$ [3], then (1) is globally asymptotically stable at the origin IOD.

Lemma 9. Efimov et al. (2016) Let $f(x_\tau) = F[x(t), x(t - \tau)]$ in (1) and the system (1) be r-homogeneous with degree $d \ge 0$ and globally asymptotically stable at the origin for $\tau = 0$, then for any $\rho > 0$ there is $0 < \tau_0 < +\infty$ such that (1) is asymptotically stable at the origin into B^τ_ρ for any delay $0 \le \tau \le \tau_0$.

Thus, (1) is locally robustly stable with respect to a sufficiently small delay if it is r-homogeneous with a nonnegative degree and stable in the delay-free case.

2.3 Local homogeneity

A disadvantage of the global homogeneity introduced so far is that such systems possess the same behavior "globally" Efimov and Perruquetti (2011), Efimov et al. (2014b).

Definition 3 Efimov and Perruquetti (2011) The function $g : C_{[-\tau,0]} \to \mathbb{R}$ is called (r,λ_0,g_0)-homogeneous ($r_i > 0$, $i = \overline{1, n}$; $\lambda_0 \in \mathbb{R} \cup \{+\infty\}$; $g_0 : C_{[-\tau,0]} \to \mathbb{R}$) if for any $\phi \in \mathcal{S}_r$ the relation

$$\lim_{\lambda\to\lambda_0} \lambda^{-d_0}g(\Lambda_r(\lambda)\phi) - g_0(\phi) = 0$$

[1] Function $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is called uniformly continuous in B^τ_ρ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, z \in B^\tau_\rho$ satisfying $||x - z||_r < \delta$, the inequality $||f(x) - f(z)||_r < \varepsilon$ holds (the homogeneous norm is used for simplicity of notation).

[2] If time is scaled $t \to \lambda^d t$ then the argument of $f : C_{[-\tau,0]} \to \mathbb{R}^n$ in (1) is also scaled to $f : C_{[-\lambda^{-d}\tau,0]} \to \mathbb{R}^n$ in (3).

[3] In this case for any $0 \le \tau \le \tau_0$, any $\varepsilon > 0$ and $\kappa \ge 0$ there is $0 \le T^\varepsilon_{\kappa,\tau} < +\infty$ such that $||x_{t,x_0}||_r \le \rho + \varepsilon$ for all $t \ge T^\varepsilon_{\kappa,\tau}$ for any $x_0 \in B^\tau_\kappa$, and $|x(t, x_0)|_r \le \sigma_\tau(||x_0||_r)$ for all $t \ge 0$ for some function $\sigma_\tau \in \mathcal{K}_\infty$ for all $x_0 \notin B^\tau_\rho$.

is satisfied (uniformly on $\mathcal{S}_r$ for $\lambda_0 \in \{0, +\infty\}$) for some $d_0 \in \mathbb{R}$.

The system (1) is called (r,λ_0,f_0)-homogeneous ($r_i > 0$, $i = \overline{1,n}$; $\lambda_0 \in \mathbb{R} \cup \{+\infty\}$; $f_0 : C_{[-\tau,0]} \to \mathbb{R}^n$) if for any $\phi \in \mathcal{S}_r$ the relation

$$\lim_{\lambda \to \lambda_0} \lambda^{-d_0} \Lambda_r^{-1}(\lambda) f(\Lambda_r(\lambda)\phi) - f_0(\phi) = 0$$

is satisfied (uniformly on $\mathcal{S}_r$ for $\lambda_0 \in \{0, +\infty\}$) for some $d_0 \geq -\min_{1 \leq i \leq n} r_i$.

For a given λ_0, g_0 and f_0 are called approximating functions.

For any $0 < \lambda_0 < +\infty$ the following formulas give an example of r-homogeneous approximating functions g_0 and f_0:

$$g_0(\phi) = ||\phi||_r^d \lambda_0^{-d_0} g(\Lambda_r(\lambda_0)\Lambda_r^{-1}(||\phi||_r)\phi),\ d \geq 0,$$
$$f_0(\phi)=||\phi||_r^d \lambda_0^{-d_0} \Lambda_r(||\phi||_r)\Lambda_r^{-1}(\lambda_0) f(\Lambda_r(\lambda_0)\Lambda_r^{-1}(||\phi||_r)\phi),$$

$d \geq -\min_{1 \leq i \leq n} r_i$. This property allows us to analyze local stability/instability of the system (1) on the basis of a simplified system

$$dy(t)/dt = f_0[y_\tau(t)], t \geq 0, \tag{4}$$

called the *local approximating dynamics* for (1).

Theorem 10. Efimov and Perruquetti (2011) Let the system (1) be (r,λ_0,f_0)-homogeneous for some $r_i > 0$, $i = \overline{1,n}$, the function f_0 be continuous and r-homogeneous with the degree d_0. Suppose there exists a locally Lipschitz continuous r-homogeneous Lyapunov-Razumikhin function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ with the degree $\nu_0 > \max\{0, -d_0\}$,

$$\alpha_1(|x|) \leq V_0(x) \leq \alpha_2(|x|)$$

for all $x \in \mathbb{R}^n$ and some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that:

(i) there exist functions $\alpha, \gamma \in \mathcal{K}$ such that for all $\varphi \in \mathcal{S}_r$

$$\max_{\theta \in [-\tau,0]} V_0[\varphi(\theta)] < \gamma\{V_0[\varphi(0)]\} \Rightarrow$$
$$D^+ V_0[\varphi(0)] f_0(\varphi) \leq -\alpha(|\varphi(0)|);$$

(ii) there exists a function $\gamma' \in \mathcal{K}$ such that $\lambda s < \gamma'(\lambda s) \leq \lambda \gamma(s)$ for all $s, \lambda \in \mathbb{R}_+ \setminus \{0\}$.

Then (the functions $\bar{\rho}_r$ and $\underline{\rho}_r$ have been defined in (2))

1) if $\lambda_0 = 0$, then there exists $0 < \bar{\lambda}_\varepsilon$ such that the system (1) is locally asymptotically stable at the origin with the domain of attraction containing the set

$$X_0 = \{\phi \in C_{[-\tau,0]} : ||\phi|| \leq \alpha_1^{-1} \circ \alpha_2 \circ \bar{\rho}_r(\bar{\lambda}_\varepsilon)\};$$

2) if $\lambda_0 = +\infty$, then there exists $0 < \underline{\lambda}_\varepsilon < +\infty$ such that the system (1) is globally asymptotically stable with respect to forward invariant set

$$X_\infty = \{\phi \in C_{[-\tau,0]} : ||\phi|| \leq \alpha_1^{-1} \circ \alpha_2 \circ \underline{\rho}_r(\underline{\lambda}_\varepsilon)\};$$

3) if $0 < \lambda_0 < +\infty$, then there exist $0 < \underline{\lambda}_\varepsilon \leq \lambda_0 \leq \bar{\lambda}_\varepsilon < +\infty$ such that the system (1) is asymptotically stable with respect to the forward invariant set X_∞ with region of attraction

$$X = \{\phi \in C_{[-\tau,0]} : \alpha_1^{-1} \circ \alpha_2 \circ \underline{\rho}_r(\underline{\lambda}_\varepsilon) < ||\phi|| < \alpha_1^{-1} \circ \alpha_2 \circ \bar{\rho}_r(\bar{\lambda}_\varepsilon)\}$$

provided that the set X is connected and nonempty.

In Efimov et al. (2014b) analysis of the input-to-state stability property for the system (1) has been presented using the homogeneity theory.

3. MAIN RESULTS

In this work we propose the following extension of Lemma 9 for the case of negative degree.

Lemma 11. Let $f(x_t) = F[x(t), x(t-\tau)]$ in (1) be uniformly continuous and the system (1) be r-homogeneous with degree $d < 0$ and globally asymptotically stable at the origin for $\tau = 0$, then for any $\varepsilon > 0$ there is $0 < \tau_0 < +\infty$ such that (1) is globally asymptotically stable with respect to B_ε^τ for any delay $0 \leq \tau \leq \tau_0$ [4].

The proof is omitted by reason of the restriction on the number of pages. Nevertheless, it should be noted that the structure of the proof of Lemma 11 is similar to the proof of the Lemma 2 in Efimov et al. (2016) for homogeneous systems with positive degree $d > 0$. However, the proof of Efimov et al. (2016) was heavily based on Lipschitz continuity property of the system, but this property is not satisfied for a negative degree and proof is mainly based on application of Lemma 1 and Lemma2.

Remark The conditions of Lemma 11 means that $F[\Lambda_r(\lambda)x, \Lambda_r(\lambda)z] = \lambda^d \Lambda_r(\lambda) F[x, z]$ for any $x, z \in \mathbb{R}^n$ and $\lambda \in (0, +\infty)$. In addition, the point $x = 0$ for the system

$$\dot{x} = F(x, x) \tag{5}$$

is globally asymptotically stable. Note that the system (5) is also homogeneous of degree $d < 0$ ($F[\Lambda_r(\lambda)x, \Lambda_r(\lambda)x] = \lambda^d \Lambda_r(\lambda) F[x, x]$ for any $x \in \mathbb{R}^n$ and $\lambda \in (0, +\infty)$), then according to Zubov (1958), Rosier (1992) there is a differentiable and r-homogeneous Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ of degree $v > -d$ such that

$$a = -\sup_{\xi \in \mathbb{S}_r} D^+ V(\xi) F(\xi, \xi) > 0,$$
$$0 < b = \sup_{|\xi|_r \leq 1} \left| \frac{\partial V(\xi)}{\partial \xi} \right| < +\infty,$$
$$c_1 = \inf_{\xi \in \mathbb{S}_r} V(\xi),\ c_2 = \sup_{\xi \in \mathbb{S}_r} V(\xi),$$
$$c_1 |x|_r^v \leq V(x) \leq c_2 |x|_r^v \quad \forall x \in \mathbb{R}^n.$$

Then the maximal possible time delay τ_0 can be estimated as follows

$$\tau_0 = \min \left\{ \frac{\bar{\rho}_r(\rho)}{\iota_k(2\bar{\rho}_r(\rho))}, \frac{1}{\pi_2(\varepsilon)} \pi^{-1} \left(\frac{\bar{\sigma}_r^{-1}(aR^{-d-v} - \epsilon)}{b} \right) \right\},$$

where $\iota_k(s) = \bar{\sigma}_r \left(k \max_{1 \leq i \leq n} [\underline{\rho}_r^{-1}(s)]^{1+d/r_i} \right)$, $R = n^{1/\rho} (\gamma c_1^{-1} c_2)^{1/v}$, $\gamma > 1$: $\sup_{\theta \in [-\tau,0]} V[\phi(\theta)] < \gamma V[\phi(0)]$

$$\pi_1(\tau) = L \max_{1 \leq i \leq n} \underline{\sigma}_r^{-1}(M\tau)^{1+d/r_i},$$
$$\pi_2(\lambda) = \begin{cases} \lambda^{-1-d/\min_{1 \leq i \leq n} r_i} & \text{if } \lambda \geq 1 \\ \lambda^{-1-d/\max_{1 \leq i \leq n} r_i} & \text{if } \lambda < 1 \end{cases}$$

$$L > 0\colon |F[\varphi(0), \varphi(-\tau)] - F[\varphi(0), \varphi(0)]|_r \leq \max\{L \max_{1 \leq i \leq n} |\varphi(0) - \varphi(-\tau)|_r^{1+d/r_i}, \eta\},$$

$$M = \sup_{|z|_r \leq \rho', |y|_r \leq \rho'} |F[z, y]|, \quad \rho' = \underline{\rho}_r^{-1}[2\bar{\rho}_r(\rho)]$$

for some $\epsilon \in \left(0, aR^{-d-v}\right)$.

[4] In this case for any $0 \leq \tau \leq \tau_0$, any $\epsilon > 0$ and $\kappa \geq \varepsilon$ there is $0 \leq T_{\kappa,\tau}^\epsilon < +\infty$ such that $||x_{t,x_0}||_r \leq \varepsilon + \epsilon$ for all $t \geq T_{\kappa,\tau}^\epsilon$ for any $x_0 \in B_\kappa^\tau$, and $|x(t, x_0)|_r \leq \sigma_\tau(||x_0||_r)$ for all $t \geq 0$ for some function $\sigma_\tau \in \mathcal{K}_\infty$ for all $x_0 \notin B_\varepsilon^\tau$.

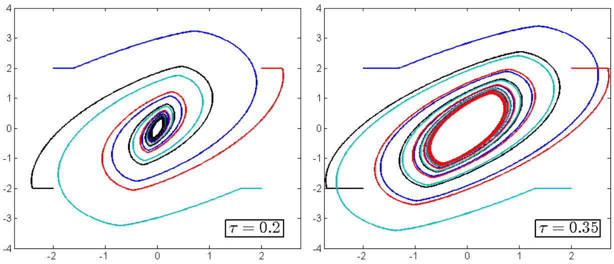

Fig. 1. Simulation results

Using Theorem 10, results of Lemma 11 can be used for local analysis of stability for not necessary homogeneous systems.

Theorem 12. Let system (1) be (r,$+\infty$,f_0)-homogeneous with degree $d_0 < 0$, $f_0(x_t) = F_0[x(t), x(t-\tau)]$ and the origin for the approximating system (4) be globally asymptotically stable for $\tau = 0$. Then (1) has bounded trajectories IOD.

Proof. The proof of Theorem 12 is a direct consequence of Theorem 10 and Lemma 11. $\square$

Roughly speaking the result of Theorem 12 says that if the approximating dynamics (4) has a negative degree and it is stable in the delay-free case, then the original system (1) has bounded trajectories for any delay.

4. NUMERICAL EXAMPLE

Consider the following system

$$\begin{cases} \dot{x}_1 = x_2 - l_1 \lfloor x_1 \rceil^{\alpha}, \\ \dot{x}_2 = -l_2 \lfloor x_1 \rceil^{2\alpha - 1}, \end{cases} \tag{6}$$

where $l_1 > 0$, $l_2 > 0$, $\alpha \in \left(\frac{1}{2}, 1\right)$ and $\lceil x \rfloor^{\beta} = |x|^{\beta}\mathrm{sign}(x)$ for any real number $\beta \geq 0$. The system (6) is homogeneous for $r = [1, \alpha]^T$ with degree $\mu = \alpha - 1 < 0$.

Let us assume that the state x_1 is available with delay $0 < \tau < +\infty$. Then according to obtained results the system (6) is globally asymptotically stable with respect to B_{ε}^{τ} for some $\varepsilon > 0$ dependent on the selected value of delay τ.

The Fig. 1 shows the simulation results for $\alpha = 0.7$, $l_1 = 1$, $l_2 = 2$ and two different values of delay τ.

5. CONCLUSIONS

For time-delay systems with negative degree it has been shown that if for zero delay the system is globally asymptotically stable, then for any delay it is converging to some compact set around the origin, that is an interesting robustness property of nonlinear homogeneous systems with respect to delays, which is not observed in the linear case. Efficiency of the proposed approach is demonstrated on example. Design of new stabilization and estimation algorithms, which preserve boundedness of the system trajectories for any delays, is a direction of future research.

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